Minimum Energy-Loss Guidance for Aeroassisted Orbital Plane Change

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Minimum energy-loss guidance for the atmospheric part of the aeroassisted plane change of an orbiting vehicle is developed and applied to the plane change of a circular orbit. First, trajectories which minimize the fuel required to change the orbital plane are computed for a realistic vehicle. From these trajectories, it is observed that the fuel weight is minimized if the velocity at exit from the atmosphere is maximized. Next, for the atmospheric turn, approximate optimal controls (angle of attack and bank angle) are found in closed form which maximize the exit velocity. Based on this approximate solution, a guidance law is developed which is implementable because only algebraic manipulations are required. Then, the minimum-fuel problem is resolved using optimal guidance for the atmospheric part of the trajectory. It is shown that this guidance law requires up to 14% more fuel than the "true" optimum for plane changes up to 40 deg.

Nomenclature

C = engine characteristic velocity (ft/s)
C_D = drag coefficient
C_{D_0} = zero-lift drag coefficient
C_D^* = drag coefficient for maximum lift-to-drag ratio
C_L = lift coefficient
C_L^* = lift coefficient for maximum lift-to-drag ratio
D = drag (lb)
H = Hamiltonian
h = altitude (ft)
K = induced drag factor
L = lift (lb)
m = mass (slugs)
p = Lagrange multiplier
r = distance from center of Earth to vehicle (ft)
S = aerodynamic reference area (ft^2)
t = time (s)
V = velocity (ft/s)
v = nondimensional velocity
w = nondimensional altitude
α = angle of attack
β = scale height (ft)
γ = flight path angle
θ = downrange angle
λ = scaled lift coefficient
μ = bank angle
$\bar{\mu}$ = gravitational constant of Earth (ft^3/s^2)
ν = multiplier in end-point function G
ρ = density (slug/ft^3)
ϕ = crossrange angle
ψ = heading angle

Subscripts

a = apogee
c = circular orbit
f = final, end of atmospheric flight
i = inflection
s = sea level
0 = initial, beginning of atmospheric flight

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Introduction

It has been known for some time that the three-impulse, aeroassisted maneuver can require less fuel to change the plane of an orbit than the single-impulse maneuver.[1] However, in order to use the aeroassisted maneuver, it is necessary to have a guidance law. In this paper, approximate analytical optimal controls are developed for the atmospheric part of the maneuver, and their use as a guidance law is demonstrated for the plane change of a circular orbit.

Early analytical work on the atmospheric segment[2] has considered constant controls and employed the assumption that gravitational and apparent lift forces are negligible with respect to aerodynamic forces. Optimal controls for the same problem are presented in Ref. 1, and their use as a guidance law in Ref. 3 shows that plane changes are limited to around 20 deg for the hypersonic vehicle used in Ref. 4. This result indicates that the gravitational and apparent lift terms are important.

In this paper, the neglected terms are included in the optimal control problem in the form of Loh's constant,[5] a technique which has been shown to yield excellent results for atmospheric entry.[6] First, optimal (minimum fuel) plane-change trajectories are computed using realistic vehicle and environment models.[4] Then approximations consistent with the optimal trajectories are introduced, and an approximate control law is developed for the atmospheric part of the maneuver.[7] A sampled-data guidance law is then developed from this control law. Finally, the minimum-fuel plane-change problem is resolved with optimal gudiance used for the atmospheric portion of the maneuver. This procedure reduces the problem to a one-dimensional minimization.

Minimum-Fuel Plane-Change Problem

The aeroassisted maneuver considered in this paper for changing the plane of a circular orbit consists of three tangential impulses and an atmospheric plane change (see Fig. 1). Deorbit is accomplished by the impulse ΔV_1 which causes the vehicle to follow an elliptic orbit to atmospheric entry. During atmospheric flight, the vehicle is flown through the plane change by modulating the angle of attack and the bank angle. Because of the loss of energy during the turn, a second impulse

ΔV_2 is required to boost the vehicle back to orbital altitude. Finally, the third impulse ΔV_3 is required to recircularize the orbit. The fuel consumed during the maneuver is represented by the sum of the three ΔV's. In this section, the minimum-fuel plane-change problem is formulated and solved as a nonlinear programming problem. This problem has been solved by a gradient projection method in Ref. 8 for a different vehicle.

Physical Model

The model used in this problem consists of the deorbit equations, the atmospheric flight equations, and the boost and reorbit equations.

Deorbit Equations

Initially, the vehicle is in circular orbit at the radius r_c and the velocity $V_c = (\bar{\mu}/r_c)^{1/2}$ where $\bar{\mu}$ is the gravitational constant of the Earth. Deorbit is accomplished by the impulse ΔV_1 which causes the vehicle to enter the atmosphere at radius r_0 with velocity V_0 and flight path inclination γ_0. For given values of r_c, V_c, ΔV_1, and r_0, the equations for conservation of energy and angular momentum lead to the following expressions for V_0 and γ_0:

$$V_0 = [(V_c - \Delta V_1)^2 - 2\bar{\mu}(1/r_c - 1/r_0)]^{1/2}$$

$$\gamma_0 = -\cos^{-1}[r_c(V_c - \Delta V_1)/r_0 V_0] \tag{1}$$

In order to ensure atmospheric entry, ΔV_1 must exceed the value

$$\Delta V_{1_{\min}} = V_c - \{2(V_c^2 - \bar{\mu}/r_0)/[1-(r_c/r_0)^2]\}^{1/2} \tag{2}$$

Atmospheric Flight Equations

The equations of motion are taken to be those of gliding flight over a nonrotating spherical Earth. If the vehicle is moving from west to east, and if a positive bank angle generates a heading toward the north, these equations are given by

$$\dot{\theta} = V\cos\gamma\cos\psi/(r\cos\phi)$$

$$\dot{\phi} = V\cos\gamma\sin\psi/r$$

$$\dot{r} = V\sin\gamma$$

$$\dot{V} = -D/m - g\sin\gamma$$

$$\dot{\gamma} = L\cos\mu/(mV) + (V/r - g/V)\cos\gamma$$

$$\dot{\psi} = L\sin\mu/(mV\cos\gamma) - (V/r)\cos\gamma\cos\psi\tan\phi \tag{3}$$

Here, θ is the longitude, ϕ the latitude, r the radial distance from the center of the Earth to the vehicle center of mass, V the velocity, γ the flight path angle, ψ the heading angle, m the mass, L the lift, D the drag, and μ the bank angle. The coordinate systems are illustrated in Fig. 2.

The acceleration of the gravity of the Earth is assumed to satisfy the inverse square law

$$g = \bar{\mu}/r^2 \tag{4}$$

and the altitude above mean sea level (defined by the radius r_s) is given by

$$h = r - r_s \tag{5}$$

The properties of the atmosphere are assumed to be functions of the altitude only. Values for the density and the speed of sound are obtained from a 1962 U.S. Standard Atmosphere.

Drag and lift are expressed as

$$D = (1/2)C_D\rho SV^2, \qquad L = (1/2)C_L\rho SV^2 \tag{6}$$

where $\rho(h)$ is the density and S is the aerodynamic reference area. The force coefficients satisfy the relations

$$C_D = C_A\cos\alpha + C_N\sin\alpha$$

$$C_L = C_N\cos\alpha - C_A\sin\alpha \tag{7}$$

where α is the angle of attack. The axial force coefficient C_A and the normal force coefficient C_N are written functionally as

$$C_A = C_A(h,M,\alpha), \qquad C_N = C_N(M,\alpha) \tag{8}$$

where $M = V/a$ is the Mach number and $a(h)$ is the speed of sound.

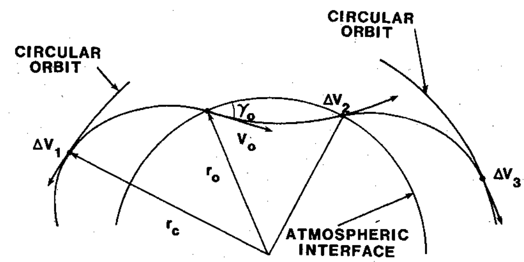

Fig. 1 Aeroassisted plane-change maneuver.

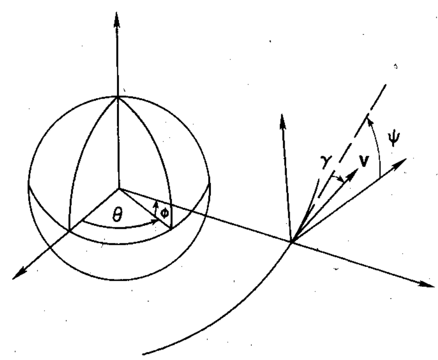

Fig. 2 Coordinate systems.

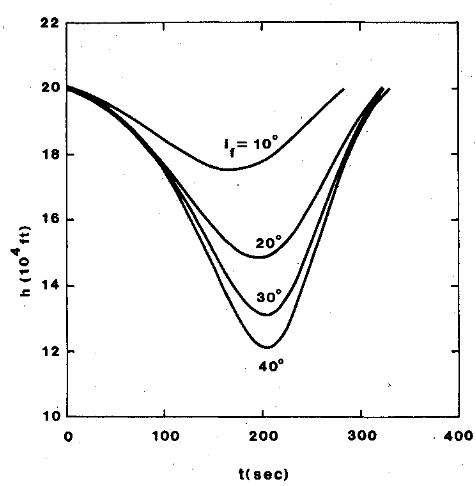

Fig. 3 "True" optimal trajectories.

The vehicle begins the maneuver with an initial mass m_c. As each impulse occurs, the mass is decreased by the amount

$$\Delta m = m_i [1 - \exp(-\Delta V/C)] \tag{9}$$

where m_i is the mass at ignition and C the characteristic velocity of the engine. Since ΔV_1 is small, the mass of the vehicle during atmospheric flight is approximately the same for all plane changes.

The orbit inclination, given by the relation

$$\cos i = \cos\phi \cos\psi \tag{10}$$

varies through the atmospheric segment and must have the required value at exit.

Finally, the vehicle used in this study is the Maneuverable Research Re-entry Vehicle (MRRV). Its aerodynamic and mass characteristics are presented in Ref. 4, along with the standard atmosphere and values for the constants. For future reference, the empty mass is 157 slugs, and the vehicle carries 179 slugs of fuel. Hence, the fuel fraction is 0.53.

Boost and Reorbit Equations

The vehicle exits the atmosphere at $r_f = r_0$, V_f, and γ_f. At this point, the impulse ΔV_2 is applied to raise the apogee of the ascending elliptical orbit to $r_a = r_c$. Conservation of energy and conservation of angular momentum lead to the following expressions for ΔV_2 and V_a (velocity at apogee):

$$\Delta V_2 = \left[\frac{2\bar{\mu}(1/r_f - 1/r_c)}{1-(r_f/r_c)^2\cos^2\gamma_f}\right]^{1/2} - V_f$$

$$V_a = \frac{r_f}{r_c}\left[\frac{2\bar{\mu}(1/r_f - 1/r_c)}{1-(r_f/r_c)^2\cos^2\gamma_f}\right]^{1/2}\cos\gamma_f \tag{11}$$

At apogee, the impulse ΔV_3 is used to increase the velocity to V_c and is given by

$$\Delta V_3 = V_c - V_a \tag{12}$$

Since r_f and r_c are fixed, it is seen that ΔV_2 and ΔV_3 are functions of only the exit conditions V_f and γ_f.

Optimal Control Problem

The minimum-fuel plane-change problem can be formulated in terms of ΔV_1 and the atmospheric turn as follows:

Find the control parameter ΔV_1 and the control histories $\alpha(t)$ and $\mu(t)$ which minimize the performance index

$$J = \Delta V_1 + \Delta V_2(V_f, \gamma_f) + \Delta V_3(V_f, \gamma_f) \tag{13}$$

subject to the differential equations of motion in Eq. (3), the initial conditions

$$t_0 = 0, \theta_0 = 0, \phi_0 = 0, r_0 \equiv \text{given}$$

$$V_0 = V_0(\Delta V_1), \gamma_0 = \gamma_0(\Delta V_1), \psi_0 = 0 \tag{14}$$

the prescribed final conditions

$$r_f = r_0, \qquad i_f \equiv \text{given} \tag{15}$$

and the constraint

$$\Delta V_1 = \Delta V_{1_{\min}} + \sigma^2 \tag{16}$$

where σ is a slack variable.

This optimal control problem is solved as a nonlinear programming problem. The final time t_f is made a parameter by introducing the transformation $\tau = t/t_f$. Also, each control function is replaced by a set of nodal points over the normalized time interval [0,1], and linear interpolation is used to form the function. Hence, the optimal control problem can be reduced to the following nonlinear programming problem: Minimize the performance index

$$J = F(X) \tag{17}$$

and subject to the equality constraints

$$C(X) = 0 \tag{18}$$

Here, the vector X contains ΔV_1, t_f, the nodal points of each control history, and σ. The performance index is the sum of the ΔV's given by Eq. (13), and the equality constraints are the two final conditions of Eq. (15) and the converted inequality constraint of Eq. (16). The equations of motion are integrated from $\tau_0 = 0$ to $\tau_f = 1$ to relate the performance index and the constraints to the parameters X.

Numerical Results

Minimum-fuel plane-change trajectories have been obtained for $i_f = 10$, 20, 30, and 40 deg. A total of twelve parameters has been used: t_f, five equally spaced nodal points for each control, and σ. The equations of motion are integrated with a fixed-step Runge-Kutta integrator using 250 integration steps. The nonlinear programming code which has been used to perform the optimization is known as VF02AD (Harwell Library), and a convergence criterion of 1.E−4 is used. All computations have been performed on the UT Cyber 170/750 computer.

The initial and final orbital altitudes are $h_c = 100$ nm. Entry into the sensible atmosphere is assumed to occur at $h_0 = 200{,}000$ ft. At this altitude, aerodynamic forces are capable of dominating gravitational and apparent lift forces.

A summary of the optimal trajectories is presented in Tables 1-4. In all cases, it is noted that the angle of attack is nearly constant at the value for maximum L/D; the bank angle initially exceeds 90 deg to pull the vehicle into the atmosphere and then decreases to pull out of the atmosphere; the latitude is negligibly small; the flight path angle is small; the inclination is approximately equal to the heading angle; and the boost impulse greatly exceeds the deorbit and reorbit impulse.

Altitude histories are shown in Fig. 3. It is seen that the minimum altitude decreases as the orbit inclination or heading angle increases.

The performance indices are summarized in Table 5, along with the fuel fractions required to make the plane changes (assuming the characteristic velocity to be $C = 10{,}000$ ft/s). The fuel fraction required to make a 40-deg plane change is 0.49. Since the total fuel fraction of the MRRV is 0.53, the highest plane change which can be made by this vehicle with

Table 1 Optimal trajectory for $i_f = 10$ deg

t, s	ϕ, deg	h, ft	V, ft/s	γ, deg	ψ, deg	α, deg	μ, deg
0	0	200000	25945	−.14	0	15.40	106.62
28.4	.010	197225	25827	−.28	.61	15.02	101.05
56.8	.043	192874	25700	−.39	1.29	14.63	95.48
85.2	.101	187422	25555	−.46	2.09	14.66	89.31
113.6	.189	181657	25372	−.44	3.11	15.10	82.53
142.0	.316	176999	25140	−.28	4.39	15.54	75.75
170.3	.487	175287	24871	.02	5.83	15.73	71.85
198.7	.705	177604	24599	.36	7.24	15.91	67.96
227.1	.962	183952	24361	.65	8.42	15.85	68.99
255.5	1.251	192302	24183	.70	9.28	15.54	74.95
283.9	1.560	199990	24054	.57	9.88	15.23	80.91

Deorbit impulse, ft/s = 124.444
Boost impulse, ft/s = 1861.279
Reorbit impulse, ft/s = 155.382
Total impulse, ft/s = 2141.105

Table 2 Optimal trajectory for $i_f = 20$ deg

t, s	ϕ, deg	h, ft	V, ft/s	γ, deg	ψ, deg	α, deg	μ, deg
0	0	200000	25942	−.23	0	15.53	108.14
32.2	.014	195166	25804	−.43	.72	14.97	103.48
64.4	.058	187615	25646	−.62	1.58	14.41	98.83
96.6	.140	177502	25444	−.79	2.72	14.21	93.18
128.8	.276	165530	25145	−.87	4.46	14.36	86.56
161.0	.494	154110	24676	−.70	7.21	14.52	79.93
193.2	.831	148341	24000	−.06	11.15	15.04	74.79
225.4	1.305	153574	23293	.85	15.19	15.55	69.26
257.6	1.881	169015	22806	1.41	17.81	15.64	69.33
289.8	2.511	186649	22547	1.27	19.11	15.30	74.95
322.0	3.165	199985	22403	.81	19.75	14.96	80.26

Deorbit impulse, ft/s = 128.025
Boost impulse, ft/s = 3478.682
Reorbit impulse, ft/s = 189.599
Total impulse, ft/s = 3796.305

Table 3 Optimal trajectory for $i_f = 30$ deg

t, s	ϕ, deg	h, ft	V, ft/s	γ, deg	ψ, deg	α, deg	μ, deg
0	0	200000	25945	−.17	0	15.56	111.21
32.2	.013	195823	25807	−.40	.70	15.06	107.35
64.4	.056	188215	25651	−.65	1.53	14.56	103.50
96.6	.137	176911	25447	−.93	2.68	14.32	98.26
128.8	.273	161894	25126	−1.18	4.57	14.35	91.62
161.0	.506	144695	24533	−1.22	8.13	14.38	84.99
193.2	.915	131826	23397	−.47	15.01	14.87	77.76
225.4	1.570	136616	22035	1.28	23.29	15.36	70.52
257.6	2.393	159343	21294	2.18	27.63	15.41	69.79
289.8	3.275	183589	21007	1.78	29.15	15.03	75.54
322.0	4.174	199987	20872	.97	29.74	14.65	81.30

Deorbit impulse, ft/s = 125.319
Boost impulse, ft/s = 4980.110
Reorbit impulse, ft/s = 219.841
Total impulse, ft/s = 5325.270

Table 4 Optimal trajectory for $i_f = 40$ deg

t, s	ϕ, deg	h, ft	V, ft/s	γ, deg	ψ, deg	α, deg	μ, deg
0	0	200000	25945	−.16	0	15.58	112.31
32.8	.014	195800	25803	−.41	.71	15.21	108.62
65.6	.059	187698	25639	−.69	1.59	14.83	104.94
98.5	.145	175158	25417	−1.03	2.84	14.59	99.72
131.3	.295	157790	25045	−1.37	5.04	14.49	92.98
164.1	.566	136932	24270	−1.51	9.74	14.39	86.23
196.9	1.087	121338	22546	−.46	20.40	14.96	78.78
229.7	1.956	130403	20615	1.92	32.76	15.54	71.34
262.5	3.004	158995	19815	2.66	37.72	15.69	69.93
295.4	4.096	185200	19548	1.88	39.16	15.44	74.56
328.2	5.198	199920	19424	.73	39.72	15.18	79.19

Deorbit impulse, ft/s = 124.845
Boost impulse, ft/s = 6469.892
Reorbit impulse, ft/s = 176.715
Total impulse, ft/s = 6771.452

this maneuver is (ignoring heating constraints) not much more than 40 deg.

Minimum Energy-Loss Atmospheric Turn

In this section, an approximate optimal trajectory is developed along the lines of Refs. 6 and 7 for the atmospheric portion of the plane-change maneuver. Furthermore, the problem is solved for arbitrary initial conditions so that the optimal controls can be used to form a feedback control law which helps compensate for the approximations.

The minimum-fuel problem can be solved in a sequential way, that is by fixing ΔV_1 and minimizing $\Delta V_2 + \Delta V_3$ with respect to t_f, $\alpha(t)$, and $\mu(t)$, and then repeating this process until the minimum value of $\Delta V_2 + \Delta V_3$ is found. Since γ is small throughout the entire optimal trajectory, $\cos\gamma_f \cong 1$ and $\Delta V_2 + \Delta V_3$ is proportional to $-V_f$. Hence, if ΔV_1 is given, the performance index in Eq. (13) is minimized by maximizing V_f or minimizing the energy-loss in the turn.

Approximate Equations of Motion

Because of the results of the previous section, it is assumed that the flight path angle is small ($\cos\gamma \cong 1$, $\sin\gamma \cong \gamma$); that the crossrange angle is sufficiently small that the term $(V/r)\cos\gamma \cos\psi \tan\phi$ can be neglected in the heading-angle equation; and that the orbit inclination is approximated by the heading angle. Next, the gravity term in the velocity equation is neglected. The atmospheric density is assumed to satisfy the exponential law

$$\rho = \rho_r \exp(-h/\beta) \tag{19}$$

where ρ_r is a reference density, and β is the constant scale height. Finally, the drag polar of the vehicle is assumed to be parabolic with constant coefficients, that is,

$$C_D = C_{D_0} + KC_L^2 \tag{20}$$

Since there are no boundary conditions imposed on the downrange angle and the crossrange angle, those equations uncouple from the system, and the relevant equations of motion reduce to

$$\begin{aligned} \dot{h} &= V\gamma \\ \dot{V} &= -C_D^*(1+\lambda^2)\rho S V^2/4m \\ \dot{\gamma} &= C_L^*\lambda\rho S V\cos\mu/2m + V[1-\bar{\mu}/V^2(r_s+h)]/(r_s+h) \\ \dot{\psi} &= C_L^*\lambda\rho S V \sin\mu/2m \end{aligned} \tag{21}$$

In these equations, the superscript * denotes the maximum lift-to-drag ratio value, that is,

$$C_L^* = (C_{D_0}/K)^{1/2} \qquad C_D^* = 2C_{D_0} \tag{22}$$

and λ is the scaled lift coefficient

$$\lambda = C_L/C_L^* \tag{23}$$

Table 5 Comparison of optimal trajectories

	"True" minimum		Guided minimum		Single impulse	
Case	Total impulse, ft/s	Fuel fraction	Total impulse, ft/s	Fuel fraction	Total impulse, ft/s	Fuel fraction
10 deg	2141	0.19	2256	0.20	4459	0.36
20 deg	3796	0.32	4313	0.35	8884	0.59
30 deg	5325	0.41	6262	0.47	13241	0.73
40 deg	6771	0.49	8147	0.56	17497	0.83

The $\dot{\gamma}$ equation can be rewritten as

$$\dot{\gamma}=(C_L^*\rho SV/2m)(\lambda\cos\mu+\bar{M}) \tag{24}$$

where

$$\bar{M}=(2m/C_L^*S)[1-\bar{\mu}/V^2(r_s+h)]/\rho(r_s+h) \tag{25}$$

The quantity $\bar{M}$ is known as Loh's constant[5] and is assumed to be constant here.

At this point, the new variables

$$w=C_L^*\rho S\beta/2m, \qquad v=2(C_L^*/C_D^*)\ln(1/V) \tag{26}$$

are introduced, and γ is chosen to be the independent variable. Hence, the approximate equations of motion for ψ, w, and v become

$$d\psi/d\gamma=\lambda\sin\mu/(\lambda\cos\mu+\bar{M})$$

$$dw/d\gamma=-\gamma/(\lambda\cos\mu+\bar{M})$$

$$dv/d\gamma=(1+\lambda^2)/(\lambda\cos\mu+\bar{M}) \tag{27}$$

Using γ as the independent variable requires care because it may not be monotonic.

Optimal Control Problem

The problem to be solved here is that of minimizing the energy loss in the atmospheric turn. It is formally stated as follows: Find the control histories $\lambda(\gamma)$ and $\mu(\gamma)$ which minimize the performance index

$$J=v_f \tag{28}$$

subject to the differential constraints in Eq. (27), the prescribed initial conditions

$$\gamma_0,\psi_0,w_0,v_0\equiv\text{given} \tag{29}$$

and the prescribed final conditions

$$\psi_f,w_f\equiv\text{given} \tag{30}$$

This problem is solved for arbitrary initial conditions so that the resulting control histories can be used in the form of a feedback control law. The development parallels that of Ref. 6.

The Hamiltonian and the augmented end-point function are given by

$$H=[p_1\lambda\sin\mu-p_2\gamma+p_3(1+\lambda^2)]/(\lambda\cos\mu+\bar{M})$$

$$G=v_f+\nu_1(\psi_f-\psi_{f_s})+\nu_2(w_f-w_{f_s}) \tag{31}$$

where p and ν denote Lagrange multipliers and the subscript s denotes a specific value. The Euler-Lagrange equations $\dot{p}=-H_x^T$ in Ref. 9 indicate that p_1, p_2, and p_3 are constants. Also, the natural boundary conditions $p_f=G_{x_f}^T$ lead to $p_3=1$.

The optimal controls are obtained from $H_\lambda=0$ and $H_\mu=0$ which, after some manipulation, lead to

$$\sin\mu=-p_1/2\lambda \tag{32}$$

and

$$\cos\mu=\bar{M}(2\lambda-p_1^2/2\lambda)/(1-\lambda^2-p_2\gamma) \tag{33}$$

Further manipulations lead to

$$\lambda^2=(1+2\bar{M}^2-p_2\gamma)\mp 2|\bar{M}|(1+\bar{M}^2-p_1^2/4-p_2\gamma)^{1/2} \tag{34}$$

$$\lambda=\{[|\bar{M}|\mp(1+\bar{M}^2-p_1^2/4-p_2\gamma)^{1/2}]^2+p_1^2/4\}^{1/2} \tag{35}$$

and

$$\lambda\cos\mu+\bar{M}=\pm(|\bar{M}|/\bar{M})(1+\bar{M}^2-p_1^2/4-p_2\gamma)^{1/2} \tag{36}$$

In these equations, $|\bar{M}|$ is needed because $\bar{M}$ can be positive or negative. Also, λ has been assumed to be ≥ 0.

Since $\dot{\gamma}$ is proportional to $\gamma\cos\mu+\bar{M}$, the following conclusions are reached relative to the signs in Eqs. (34) through (36):

$$\begin{aligned}&\dot{\gamma}>0,\bar{M}>0; &&\text{upper sign}\\ &\dot{\gamma}>0,\bar{M}<0; &&\text{lower sign}\\ &\dot{\gamma}<0,\bar{M}>0; &&\text{lower sign}\\ &\dot{\gamma}<0,\bar{M}<0; &&\text{upper sign}\end{aligned} \tag{37}$$

If $\dot{\gamma}$ changes sign, Eq. (36) requires that

$$\gamma_i=(1+\bar{M}^2-p_1^2/4)/p_2 \tag{38}$$

at an inflection point ($\dot{\gamma}=0$). Since the multipliers are constant, there is only one value of γ_i. Hence, at most one inflection point can exist in a trajectory.

At this point, no condition has been imposed on γ_f so that the natural boundary condition $H_f=-G_{\gamma_f}=0$ must be satisfied. By using Eqs. (34) through (36), the Hamiltonian at

Table 6 Optimal guided trajectory for $\psi_f=10$ deg

t, s	ϕ, deg	h, ft	V, ft/s	γ, deg	ψ, deg	α, deg	μ, deg
0.0	0.000	200000	25943	−.21	0.00	15.10	88.82
40.0	.023	196658	25759	−.16	.97	15.11	85.42
80.0	.096	194486	25556	−.08	2.04	15.15	82.79
120.0	.221	193642	25343	−.01	3.15	15.20	81.57
160.0	.398	193872	25128	.04	4.27	15.28	80.05
200.0	.626	194811	24919	.07	5.35	15.40	78.25
240.0	.902	196149	24717	.08	6.38	15.58	76.18
280.0	1.221	197561	24523	.08	7.33	15.81	73.95
320.0	1.580	198811	24337	.06	8.24	16.10	71.76
360.0	1.976	199683	24154	.04	9.10	16.43	69.79
400.0	2.406	199999	23971	.00	9.93	16.78	67.55

Deorbit impulse, ft/s = 126.882
Boost impulse, ft/s = 2007.022
Reorbit impulse, ft/s = 122.318
Total impulse, ft/s = 2256.222

Table 7 Optimal guided trajectory for $\psi_f=20$ deg

t, s	ϕ, deg	h, ft	V, ft/s	γ, deg	ψ, deg	α, deg	μ, deg
0	0	200000	25852	−.97	0	15.15	82.23
40.0	.028	183966	25628	−.79	1.27	15.16	79.99
80.0	.136	172473	25255	−.49	3.36	15.17	79.16
120.0	.355	166951	24792	−.15	5.96	15.20	78.49
160.0	.696	167054	24301	.15	8.72	15.24	77.80
200.0	1.150	171109	23854	.32	11.22	15.35	76.23
240.0	1.700	177237	23454	.40	13.42	15.63	73.97
280.0	2.324	183818	23127	.40	15.13	16.22	69.80
320.0	3.005	189790	22848	.34	16.50	17.18	64.99
360.0	3.729	194702	22593	.27	17.63	18.46	60.95
400.0	4.489	198235	22346	.17	18.64	19.82	58.81
452.0	5.521	199997	22017	.00	19.89	21.53	55.59

Deorbit impulse, ft/s = 220.063
Boost impulse, ft/s = 3971.266
Reorbit impulse, ft/s = 122.320
Total impulse, ft/s = 4313.649

the final point can be rewritten as

$$H_f = 2[-(1+\bar{M}^2 - p_1^2/4 - p_2\gamma_f)^{1/2} \pm |\bar{M}|]/(\pm|\bar{M}|/\bar{M}) \quad (39)$$

and the boundary condition becomes

$$\gamma_f = (1 - p_1^2/4)/p_2 \quad (40)$$

However, $H_f = 0$ can only be satisfied under the following conditions:

$$\bar{M} > 0, \quad \dot{\gamma}_f > 0, \quad \text{and} \quad \bar{M} < 0, \quad \dot{\gamma}_f < 0 \quad (41)$$

A consequence of these results is that Eqs. (36) and (40) lead to

$$\lambda_f = -p_1/2, \qquad \mu_f = \pi/2 \quad (42)$$

In order to evaluate p_1, p_2, and γ_f, it is necessary to determine all possible geometries of the trajectory, develop the equations to be solved for each geometry, solve each set of equations, and pick the set of constants which gives the lowest value of v_f. There can be trajectories with no inflection points, and there can be trajectories with one inflection point, but Eq. (41) must be satisfied at the final point.

At this point, the equations of motion, Eq. (27), are combined with Eqs. (32), (34), and (36) and integrated for all possible combinations of subarcs. In doing so, the following definitions are introduced:

$$a = 1 + \bar{M}^2 + P_1^2/4 \quad (43)$$

$$b = 1 + \bar{M}^2 - p_1^2/4 \quad (44)$$

Since

$$b - p_2\gamma_i = 0 \quad (45)$$

at the inflection point, the integration of Eq. (43) for all possible trajectories leads to

$$\psi_f - \psi_0 = \mp(\bar{M}/|\bar{M}|)(p_1/p_2)[(b-p_2\gamma_f)^{1/2} \pm (b-p_2\gamma_0)^{1/2}] \quad (46)$$

$$w_f - w_0 = \pm(\bar{M}/|\bar{M}|)(2/3p_2^2)[(2b+p_2\gamma_f)(b-p_2\gamma_f)^{1/2} \pm (2b+p_2\gamma_0)(b-p_2\gamma_0)^{1/2}] \quad (47)$$

$$v_f - v_0 = -2\bar{M}(\gamma_f-\gamma_0) \mp (\bar{M}/|\bar{M}|)(2/p_2)\{[a(b-p_2\gamma_f)^{1/2} + (1/3)(b-p_2\gamma_f)^{3/2}] \pm [a(b-p_2\gamma_0)^{1/2} + (1/3)(b-p_2\gamma_0)^{3/2}]\} \quad (48)$$

In these equations, the first set of $\pm$ signs refers to $\dot{\gamma}_f > 0$ (upper sign) and $\dot{\gamma}_f < 0$ (lower sign), while the second set of $\pm$ signs indicates one inflection (upper sign) or no inflection (lower sign).

Equations (40), (46), and (47) form a system of three equations in three unknowns p_1, p_2, and γ_f. Given values for $\bar{M}$, ψ_0, ψ_f, w_0, w_f, and the signs, these equations are solved by Newton's method. Then, given v_0, the set which gives the lowest value of v_f defines the optimal trajectory. Once the multipliers are known, the optimal controls are obtained from Eqs. (32), (33), and (34).

For some boundary conditions, trajectories have been found which end with $\gamma_f < 0$. This means that the ascent trajectory would pass through the atmosphere while returning to orbit and violates the assumption of no atmospheric forces. For these paths, the condition $\gamma_f = 0$ is imposed. Here, the above procedure is repeated with just Eqs. (46) and (47).

Table 8 Optimal guided trajectory for $\psi_f = 30$ deg

t, s	φ, deg	h, ft	V, ft/s	γ, deg	ψ, deg	α, deg	μ, deg
0	0	200000	25716	−1.52	0	15.22	79.22
40.0	.032	174765	25453	−1.27	1.56	15.18	78.93
80.0	.175	156386	24901	−.78	4.75	15.16	79.15
120.0	.502	148254	24070	−.17	9.62	15.15	79.53
160.0	1.048	149895	23190	−.33	14.87	15.20	78.39
200.0	1.777	157297	22485	.57	19.10	15.39	75.78
240.0	2.630	166727	21962	.62	22.16	15.97	70.85
280.0	3.566	176015	21550	.61	24.41	16.93	64.38
320.0	4.558	184740	21196	.55	26.14	18.86	58.09
360.0	5.589	192232	20863	.46	27.54	21.62	53.49
400.0	6.649	197886	20523	.30	28.77	24.43	53.60
437.7	7.669	200000	20186	.01	29.94	28.04	49.96

Deorbit impulse, ft/s = 350.099
Boost impulse, ft/s = 5781.665
Reorbit impulse, ft/s = 122.330
Total impulse, ft/s = 6262.291

Table 9 Optimal guided trajectory for $\psi_f = 40$ deg

t, s	φ, deg	h, ft	V, ft/s	γ, deg	ψ, deg	α, deg	μ, deg
0	0	200000	25701	−1.57	0	15.19	80.82
40.0	.032	173528	25433	−1.36	1.60	15.14	80.61
80.0	.182	153020	24841	−.93	5.07	15.13	80.53
120.0	.541	142331	23850	−.30	10.96	15.12	80.85
160.0	1.172	142345	22712	.26	17.92	15.17	80.25
200.0	2.034	148701	21800	.51	23.64	15.36	77.44
240.0	3.049	156955	21137	.59	27.76	15.84	71.87
280.0	4.160	165704	20620	.60	30.81	16.93	65.21
320.0	5.335	174171	20179	.60	33.16	18.45	58.78
360.0	6.553	182452	19758	.59	35.12	21.32	53.07
400.0	7.799	190217	19320	.55	36.81	25.53	49.06
440.0	9.062	196883	18831	.43	38.38	30.36	49.05
479.0	10.303	200000	18308	.00	39.96	35.63	51.90

Deorbit impulse, ft/s = 373.911
Boost impulse, ft/s = 7650.863
Reorbit impulse, ft/s = 122.319
Total impulse, ft/s = 8147.093

Optimal Guidance

In the next section, the minimum-fuel problem is solved with the atmospheric turn replaced by optimal guidance. It is assumed that the MRRV can provide γ_0, ψ_0, h_0, and V_0 at each sample time which is taken to be 1 s. Approximate optimal values are computed for C_{L_0} and μ_0 and are held constant over the sample period. An iteration process is needed to relate C_L to the angle of attack. In addition, the value of $\bar{M} = \bar{M}(h, V)$ is updated at each sample time.

The constants needed for the exponential atmosphere are obtained by fitting the exponential function, Eq. (19), to the standard atmosphere densities at $h_1 = 180{,}000$ ft and $h_2 = 200{,}000$ ft. Hence,

$$\beta = (h_2 - h_1)\ln(\rho_2/\rho_1)$$

$$\rho_r = \rho_2 \exp(h_2/\beta) \quad (49)$$

Finally, the drag polar constants for the MRRV are obtained from vehicle data at $h = 200{,}000$ ft and are given by

$$C_{D_0} = 0.032, \qquad K = 1.4 \quad (50)$$

meaning that the maximum lift-to-drag ratio is $E^* = 2.4$.

Minimum Fuel Trajectories with Optimal Guidance

If the atmospheric portion of the plane-change maneuver is replaced by optimal guidance, the minimum-fuel problem

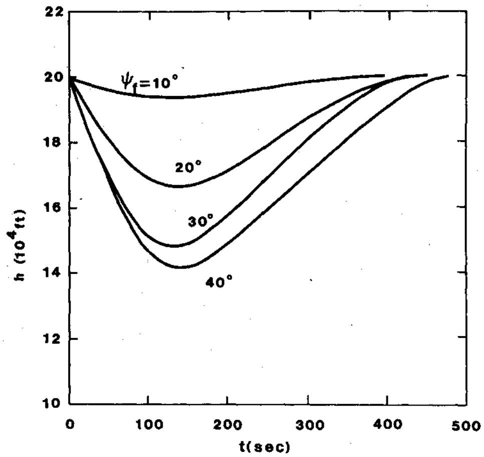

Fig. 4 Guided optimal trajectories.

reduces to a one-dimensional minimization problem, the parameter being ΔV_I. Results have been obtained for 10, 20, 30, and 40 deg turns. The guided turn is ended when the exit altitude $h_f = 200{,}000$ ft is reached. Any deficiency in ψ_f is made up by a ΔV which is added to the performance index.

Values of the performance index and the fuel fraction needed to make the plane change are presented in Table 5. The fuel fraction is computed with the formula

$$\Delta m/m_c = 1 - \exp(-\Delta V/C) \tag{51}$$

Also contained in Table 5 are the corresponding values for the "true" optimal trajectories and those for the single-impulse plane change. Values for the latter are

$$\Delta V = 2V_c \sin(\Delta i/2) \tag{52}$$

Table 5 shows that the optimal-guidance trajectories require a larger ΔV than the true optimal trajectories but require less than one-half the ΔV needed by the single-impulse maneuver. However, this vehicle cannot make a 40-deg plane change because the fuel required is greater than the amount available (fuel fraction = 0.53).

Characteristics of the resulting trajectories are listed in Tables 6-9. Compared with the "true" optimal trajectories, the optimal guided trajectories require about 50% more time in the atmosphere, but they do not descend as deeply into the atmosphere. Hence, while the maximum heat transfer rate will be less, the heat load will be greater. The optimal guided trajectories also have higher entry angles and lose more velocity (as expected). The additional velocity loss can be attributed to the increase in angle of attack near the end of the path which, in turn, is related to high negative values of $\dot{M}$.

Altitude histories are presented in Fig. 4.

Discussion and Conclusions

The minimum-fuel orbit plane-change problem has been solved by nonlinear programming for the plane change of a circular orbit. Approximations consistent with the optimal trajectories have been used to develop analytical controls for the atmospheric portion of the plane-change maneuver. Finally, a feedback controller based on repeated application of the approximate optimal controls is used to replace the atmospheric portion, and minimum-fuel trajectories are recalculated. In general, an optimal guided trajectory requires more fuel (up to 14% more) than a "true" optimal trajectory. However, it is possible to implement the analytical guidance law because only algebraic manipulations are required. Finally while the use of minimum energy-loss guidance has been demonstrated only for a circular orbit plane-change, the guidance law is valid for the atmospheric part of any plane-change maneuver.

Since the atmospheric part of the approximate plane-change trajectory must go deep enough into the atmosphere for aerodynamic forces to be able to dominate gravitational and apparent lift forces, improvements in the approximation can be made by matched asymptotic approximations between the atmospheric arc and the Keplerian arc.[10]

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